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On Euler's magic matrices of sizes 3 and 8

by

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Abstract. A proper Euler's magic matrix is an integer $n \times n$ matrix $M \in \mathbb{Z}^{n\times n}$ such that $M \cdot M^t = \gamma \cdot I$ for some nonzero constant γ, the sum of the squares of the entries along each of the two main diagonals equals γ, and the squares of all entries in M are pairwise distinct. Euler constructed such matrices for $n = 4$. In this work, we use multiplication matrices of the octonions to construct examples for $n = 8$, and prove that no such matrix exists for $n = 3$.

1. Introduction. A classical magic square is an $n \times n$ matrix A with distinct nonnegative integer entries such that the sums of the entries in each row, each column, and both main diagonals are the same.

If one requires in addition that the entries of A are square numbers, then A is called a *magic square of squares*; see [2, 14]. It is an open problem whether a magic square of squares of size 3 exists, despite considerable effort on this question in [3, 4].

Leonhard Euler looked at the question for $n = 4$. He noticed that orthogonal matrices, or slightly more general matrices M such that $M \cdot M^t = \gamma \cdot I$, could make the problem easier, for if A is the matrix whose entries are the squares of those of M, then the conditions on the row and column sums for A are automatically fulfilled. So one is faced with only two polynomial conditions for the two main diagonals, and the added requirement that the entries of A are pairwise distinct.

DEFINITION 1.1. Let R be a commutative ring and n be a positive integer. A matrix $M = (m_{i,j})_{1\le i,j\le n} \in R^{n\times n}$ is called an *Euler's magic matrix* over R if the following holds for some $\gamma \in R \setminus \{0\}$, where I_n denotes the $n \times n$

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identity matrix:

$$M \cdot M^t = \gamma \cdot I_n, \tag{1}$$

$$\sum_{i=1}^{n} m_{i,i}^2 = \gamma, \tag{2}$$

$$\sum_{i=1}^{n} m_{i,n+1-i}^2 = \gamma. \tag{3}$$

If in addition the squares of the entries in M are pairwise distinct, then we call M a *proper Euler's magic matrix.*

Note that the sum in (3) is the sum of the squares of the elements on the anti-diagonal. As remarked above, if M is a proper Euler's magic matrix, then $A = (a_{i,j})_{1\le i,j\le n}$ with $a_{i,j} = m_{i,j}^2$ is a magic square of squares.

The case $n = 2$ is easy to work out: the only Euler's magic 2×2 matrices have the form

$$\begin{pmatrix} a & a \\ a & -a \end{pmatrix}, \quad \begin{pmatrix} a & a \\ -a & a \end{pmatrix}, \quad \begin{pmatrix} a & -a \\ a & a \end{pmatrix}, \quad \begin{pmatrix} -a & a \\ a & a \end{pmatrix}.$$

In particular, there are no proper such cases.

Euler studied the intriguing problem of whether a proper Euler's magic matrix exists for $n = 4$, referring to it as a "problema curiosum". In fact, he managed to produce a multi-parameter family of such matrices. A particular case, due to Euler too, is

$$M = \begin{pmatrix} 68 & -29 & 41 & -37 \\ -17 & 31 & 79 & 32 \\ 59 & 28 & -23 & 61 \\ -11 & -77 & 8 & 49 \end{pmatrix}.$$

A more systematic study for the case $n = 4$, based on the algebra of quaternions, was given by Hurwitz [7, Vorlesung 12]. See [2], [13] and in particular [10] and [11, Lecture 12] for the history of this problem and its connection with the Hamilton quaternions.

In Section 2 we show this negative result concerning 3×3 matrices:

Theorem 1.2. *There is no Euler's magic matrix in $\mathbb{Q}^{3\times 3}$.*

It is perhaps surprising that $n = 3$ is the only positive integer for which there is no Euler's magic matrix in $\mathbb{Q}^{n\times n}$. In fact, in Section 4 we provide a simple construction of Euler's magic matrices in $\mathbb{Z}^{n\times n}$ for each $n \neq 3$. However, these matrices are far from proper. In particular, we do not know whether a proper Euler's magic matrix exists for $n = 5$. Section 4 contains some examples which come close.

In Section 3 we examine the case $n = 8$. In [12, 13], Ísabel Pirsic suggested using certain matrices coming from left and right multiplication of the octonions to find a construction analogous to Euler's for $n = 4$. However, a massive search by her did not bring an example. In this note, we demonstrate that a careful analysis of the polynomial system yields solutions without much searching. So Pirsic's suggestion to use octonion multiplication matrices proved helpful.

In order to describe our results (and later the methods), we introduce the matrices

$$L(a,b,\dots,h) = \begin{pmatrix} a & -b & -c & -d & -e & -f & -g & -h \\ b & a & -d & c & -f & e & h & -g \\ c & d & a & -b & -g & -h & e & f \\ d & -c & b & a & -h & g & -f & e \\ e & f & g & h & a & -b & -c & -d \\ f & -e & h & -g & b & a & d & -c \\ g & -h & -e & f & c & -d & a & b \\ h & g & -f & -e & d & c & -b & a \end{pmatrix}$$

and

$$R(p,q,\dots,w) = \begin{pmatrix} p & -q & -r & -s & -t & -u & -v & -w \\ q & p & s & -r & u & -t & -w & v \\ r & -s & p & q & v & w & -t & -u \\ s & r & -q & p & w & -v & u & -t \\ t & -u & -v & -w & p & q & r & s \\ u & t & -w & v & -q & p & -s & r \\ v & w & t & -u & -r & s & p & -q \\ w & -v & u & t & -s & -r & q & p \end{pmatrix}.$$

With respect to a suitable basis of the octonions $\mathbb{O}$ over the real numbers, $L(a,b,\dots,h)$ describes the left multiplication $\mathbb{O} \to \mathbb{O}$, $z \mapsto xz$, where x has the coefficients $a,b,\dots,h$. Likewise, $R(p,q,\dots,w)$ describes the right multiplication $\mathbb{O} \to \mathbb{O}$, $z \mapsto zx$, where x has coefficients $p,q,\dots,w$.

The reason that we use different letters for the entries of $L(\cdot)$ and $R(\cdot)$ is that the matrices we will study have the form $M = L \cdot R = L(a,b,\dots,h) \cdot R(p,q,\dots,w)$. The point is that for arbitrary $a,b,\dots,h,p,q,\dots,w$ we have $L \cdot L^t = (a^2 + b^2 + \cdots + h^2) \cdot I_8$ and $R \cdot R^t = (p^2 + q^2 + \cdots + w^2) \cdot I_8$, and therefore $M \cdot M^t = \gamma \cdot I_8$ with $\gamma = (a^2 + b^2 + \cdots + h^2)(p^2 + q^2 + \cdots + w^2)$.

Thus condition (1) in Definition 1.1 is automatically satisfied, and one has "only" to discuss the two polynomial conditions (2) and (3) in the 16 unknowns $a, b, \dots, h, p, q, \dots, w$, and the properness. A typical example that we obtain is

Theorem 1.3. *Set*

$$L = L(2,\ 1,\ 1,\ 4,\ 2,\ 1,\ 1,\ -2),$$
$$R = R(-7,\ -55,\ -11,\ 1,\ -27,\ -13,\ -19,\ 4).$$

Then

$$M = L \cdot R = \begin{pmatrix} 142 & 197 & -225 & 30 & 16 & 57 & -13 & -170 \\ -37 & -60 & 136 & 201 & 177 & 98 & -32 & -193 \\ -283 & -4 & -148 & -95 & 71 & 164 & 10 & -1 \\ -22 & 237 & 181 & -178 & 138 & -29 & -45 & -8 \\ -120 & 97 & 27 & 74 & -62 & -129 & 293 & -82 \\ -9 & 38 & -116 & 131 & 235 & -144 & 0 & 187 \\ -103 & 180 & 50 & 195 & -163 & 64 & -132 & 107 \\ -126 & -35 & -51 & -20 & -42 & -247 & -191 & -154 \end{pmatrix}$$

is a proper Euler's magic matrix.

This single example arises from specializing the parameters of a 4-parameter family of Euler's magic matrices to values which preserve properness. In this case, the example was obtained from setting $(q, r, t, u) = (-55, -11, -27, -148)$ and rescaling in the following theorem.

Theorem 1.4. *For variables q, r, t, u over $\mathbb{Q}$ set*

$$X = 7q^2 + 7r^2 + 21qt - 7rt + 34t^2 - 7qu - 21tu + 4u^2 + 7q + 21r - 7u + 34$$

and

$$L = L(2,\ 1,\ 1,\ 4,\ 2,\ 1,\ 1,\ -2),$$
$$R = R\left(\frac{3(t^2-1)u}{2X},\ q,\ r,\ 1,\ t,\ u-q-3t-1,\ t-r-3,\ \frac{u^2-X}{2u}\right).$$

Then $M = L \cdot R$ is a proper Euler's magic matrix over $\mathbb{Q}(q, r, t, u)$.

Remark 1.5. The reader who wishes to verify the examples need not type these matrices. A proof is provided in the ancillary SageMath script `euler_verify.sage` at [9]. This script can be run at the SageMathCell at https://sagecell.sagemath.org/.

2. There are no Euler's magic 3×3 matrices. If we look for Euler's magic $n \times n$ matrices for odd n over a field, then we may assume that these matrices are orthogonal:

LEMMA 2.1. *Let K be a field, n be odd, and $M \in K^{n\times n}$ with $M \cdot M^t = \gamma \cdot I_n$ for $0 \neq \gamma \in K$. Then $\gamma = \lambda^2$ for $\lambda \in K$, therefore $\frac{1}{\lambda} \cdot M$ is orthogonal.*

Proof. Write $n = 2k+1$. Taking the determinant of $M \cdot M^t = \gamma \cdot I_n$ yields $(\det M)^2 = \gamma^n = \gamma^{2k+1}$, so $\gamma = \lambda^2$ for $\lambda = \det M/\gamma^k$. ∎

We use the following refinement of the Cayley transform which parametrizes orthogonal real matrices:

PROPOSITION 2.2. *Let $M \in \mathbb{R}^{n\times n}$ be an orthogonal matrix. Then one can write*

$$DM = (I_n - S)(I_n + S)^{-1},$$

where $S \in \mathbb{R}^{n\times n}$ is skew-symmetric and $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix with diagonal entries in $\{-1, 1\}$.

Note that 0 is the only possible real eigenvalue of a real skew-symmetric matrix, hence $I_n + S$ is invertible for every real skew-symmetric matrix S.

REMARK 2.3. This proposition was stated and proved (in a slightly different form) in 1991 by Liebeck and Osborne [8]. However, it had already appeared 30 years earlier in [1, Chapter 6, Section 4, Exercises 7–11]. We briefly sketch the argument: The Cayley transform $S \mapsto M = (I_n - S)(I_n + S)^{-1}$ is a bijection from the set of skew-symmetric matrices $S \in \mathbb{R}^{n\times n}$ to the set of orthogonal matrices $M \in \mathbb{R}^{n\times n}$ for which -1 is not an eigenvalue. This map is involutive in the sense that if $M = (I_n - S)(I_n + S)^{-1}$, then $S = (I_n - M)(I_n + M)^{-1}$. All of this has been well known since Cayley's time (and is easy to verify). To prove the proposition, one needs to find D such that -1 is not an eigenvalue of DM. Since $DM + I_n = D \cdot (M + D)$, this is equivalent to $M + D$ being invertible. But the existence of D follows from an easy induction on n (for arbitrary $M \in \mathbb{R}^{n\times n}$); see, e.g., [6, Lemma 1].

We now prove Theorem 1.2. Let $M \in \mathbb{Q}^{3\times 3}$ be an Euler's magic matrix. Multiplying M by a nonzero rational preserves this property, so in view of Lemma 2.1 we may assume that M is orthogonal. Moreover, the property of M being an Euler's magic matrix is preserved upon replacing rows by their negatives. Thus, by Proposition 2.2, we may assume that

$$M = (I_3 - S)(I_3 + S)^{-1}, \quad \text{where} \quad S = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \in \mathbb{Q}^{3\times 3}.$$

With $\varDelta = \det(I_3 + S) = a^2 + b^2 + c^2 + 1$ we compute

$$M = \frac{1}{\varDelta} \begin{pmatrix} -a^2 - b^2 + c^2 + 1 & -2bc - 2a & 2ac - 2b \\ -2bc + 2a & -a^2 + b^2 - c^2 + 1 & -2ab - 2c \\ 2ac + 2b & -2ab + 2c & a^2 - b^2 - c^2 + 1 \end{pmatrix}.$$

The conditions (2) and (3) about the diagonal and the anti-diagonal are $D = E = 0$ with

$$\begin{aligned}D &= (-a^2 - b^2 + c^2 + 1)^2 + (-a^2 + b^2 - c^2 + 1)^2 + (a^2 - b^2 - c^2 + 1)^2 - \Delta^2 \\ &= 2(a^4 - 2a^2b^2 + b^4 - 2a^2c^2 - 2b^2c^2 + c^4 - 2a^2 - 2b^2 - 2c^2 + 1)\end{aligned}$$

and

$$\begin{aligned}E &= (2ac - 2b)^2 + (-a^2 + b^2 - c^2 + 1)^2 + (2ac + 2b)^2 - \Delta^2 \\ &= 4(-a^2b^2 + 2a^2c^2 - b^2c^2 - a^2 + 2b^2 - c^2).\end{aligned}$$

Now, while D and E look somewhat complicated, we get by some magical calculation

$$\frac{D + E}{2} = (a^2 - 2b^2 + c^2 - 2)^2 - 3(b^2 + 1)^2.$$

From $D = E = 0$ and the fact that 3 is not a square in $\mathbb{Q}$ we get $b^2 + 1 = 0$, a contradiction.

Remark 2.4. Reducing to orthogonal matrices and using the Cayley transform is quite natural and straightforward. But how could one have guessed that $D = E = 0$ with D and E as above has no rational solution? We sketch another, less "magical", proof.

The first thing one might observe is that D and E are polynomials in a^2, b^2, and c^2, and that D and E are symmetric in a^2 and c^2. Thus we can express D and E in terms of $\beta = b^2$, $s = a^2 + c^2$, and $p = a^2c^2$. We obtain

$$\begin{aligned}D/2 &= \beta^2 - 2(1 + s)\beta + (1 - s)^2 - 4p, \\ E/4 &= (2 - s)\beta - s + 2p.\end{aligned}$$

Eliminating β from $D = E = 0$ yields

$$\begin{aligned}0 &= 4p^2 + \bigl(-8s^2 + 16s - 8\bigr)\, p + s^4 - 4s^3 + 12s^2 - 16s + 4 \\ &= 4(p - (s - 1)^2)^2 - 3(s - 2)^2 s^2.\end{aligned}$$

As 3 is not a square in $\mathbb{Q}$, we get $s = 0$ or $s = 2$, and so $p = (s-1)^2 = 1$. The case $s = 0$ yields $a^2 + c^2 = 0$. Then $a = c = 0$, and therefore $p = a^2c^2 = 0$, a contradiction.

In the other case we have $a^2 + c^2 = s = 2$ and $a^2c^2 = p = 1$, hence

$$(a^2 - 1)^2 + (c^2 - 1)^2 = (a^2 + c^2)^2 - 2(a^2 + c^2) + 2 - 2a^2c^2 = 0.$$

We get $a^2 = c^2 = 1$. This yields $D/4 = b^4 - 6b^2 - 3 = (b^2 - 3)^2 - 12$, but there is no $b \in \mathbb{Q}$ with $D = 0$.

3. Euler's magic 8×8 matrices. We look for 16 integers $a, b, \ldots, h$, $p, q, \ldots, w$ such that $M = L \cdot R$ with $L = L(a, b, \ldots, h)$ and $R = R(p, q, \ldots, w)$ is a proper Euler's magic matrix.

As remarked in the introduction, we have

$$M \cdot M^t = (a^2 + b^2 + \cdots + h^2)(p^2 + q^2 + \cdots + w^2)I_8,$$

so (1) in Definition 1.1 holds with $\gamma = (a^2+b^2+\cdots+h^2)(p^2+q^2+\cdots+w^2)$. Thus the conditions (2) and (3) have to be studied. In the 3×3 case from the previous section, it turns out that actually the sum of the equations (2) and (3) has a useful property. Here again, it appears that the sum and the difference of (2) and (3) have somewhat better properties.

Accordingly, with $M = (m_{i,j})_{1 \le i,j \le 8}$, set

$$A(a,b,\ldots,w) = \sum_{i=1}^{8} m_{i,i}^2 - \sum_{i=1}^{8} m_{i,9-i}^2,$$
$$B(a,b,\ldots,w) = \sum_{i=1}^{8} m_{i,i}^2 + \sum_{i=1}^{8} m_{i,9-i}^2 - 2\gamma.$$

Thus M is an Euler's magic matrix if and only if $A(a,b,\ldots,w) = 0$ and $B(a,b,\ldots,w) = 0$.

3.1. Some properties of $A(a,b,\ldots,w)$ and $B(a,b,\ldots,w)$. The strategy is the following. We fix integers $a,b,\ldots,h$ and consider $p,q,\ldots,w$ as variables. Then each entry of $M = (m_{i,j})$ is a linear form in $p,q,\ldots,w$, and therefore A and B are homogeneous quadratic forms.

To ease the language, we call an arbitrary matrix *proper* if the squares of its entries are pairwise distinct.

Also, it is obvious that if a matrix which depends on parameters is not proper, then this is even more true if we specialize parameters.

So in order to start, we need to choose $a,b,\ldots,h \in \mathbb{Z}$ such that $M \in \mathbb{Z}[p,q,\ldots,w]^{8\times 8}$ is proper.

This for instance requires that $a \neq 0$ or $h \neq 0$, as $m_{1,8} - m_{8,1} = 2(aw - hp)$. In fact, if two of the numbers $a,b,\ldots,h$ vanish, then M is not proper, as one can easily check with a simple program.

But certain other choices of $a,b,\ldots,h$ are seen to be impossible even if M is proper. For instance, M is proper for $a = b = c = d = e = f = g = h = 1$. But in this case we get

$$A(p,q,\ldots,w) = (p+q+t+u)(r+s+v+w),$$

which forces $p+q+t+u = 0$ or $r+s+v+w$. However,

$$m_{3,3} - m_{3,6} = 2(p+q+t+u),$$
$$m_{2,2} - m_{2,7} = 2(r+s+v+w),$$

so no matter which factor vanishes, we see that the condition $A(p,q,\ldots,w) = 0$ forces M to be improper. The same happens (for other index pairs of M) whenever all $a,b,\ldots,h$ are in $\{-1,1\}$.

However, if at least one of the integers $a, b, \dots, h$ is different from ± 1 (and assuming without loss of generality that $\gcd(a, b, \dots, h) = 1$), it rarely happens that the quadratic form A (or B) is reducible.

A necessary condition for $A = B = 0$ of course is that the quadratic forms A and B are isotropic, and in fact that every linear combination $\lambda A + \mu B$ is isotropic. However, that is a very weak condition, because we have more than four variables, so the only condition is that $\lambda A + \mu B$ is isotropic over $\mathbb{R}$. But for random choices of $a, b, \dots, h$ this is usually the case.

3.2. A naive search. If we eliminate one of the variables, w say, from $A = B = 0$, then in general we will obtain a quartic form in $p, q, \dots, v$. Next, if we specialize all but two of these variables to integers, then usually the resulting curve in the remaining variables will be quartic. Many experiments have shown that the rational points of this quartic are hard to analyze. Despite having degree 4, its genus (if it is absolutely irreducible) will be at most 1. (It is a known fact from algebraic geometry that if the intersection of two quadratic surfaces in $\mathbb{C}^3$ is an irreducible curve, then its genus is at most 1; see, e.g., [5, Lecture 22, Pencils of Quadrics].) However, in general the curve will have no rational singularities which could help to transform it into a cubic. But even in cases when there were rational singularities, and furthermore the transformed cubic could be transformed into Weierstrass normal form, the software like SageMath, Pari, or Magma was not able to compute the Mordell–Weil rank of these curves in all cases we tried. The easily computed torsion points led to no examples. Note that even if we have a rational point on the curve, then w usually has degree 2 over $\mathbb{Q}$, because in general A has w-degree 2. And in most cases where we found rational solutions of $A = B = 0$, the resulting matrix M was not proper.

In rare cases, however, we found valid solutions by this approach. For instance, for

$$(a, b, \dots, h) = (0, 1, 1, 1, 1, 1, -1, 5),$$
$$(p, q, r, s, t) = (3, -2, -4, 5, 6),$$

the resulting system $A(u, v, w) = B(u, v, w) = 0$ has the rational solution $(u, v, w) = (13/15, -14/15, -23/5)$, which indeed gives, after rescaling, the proper Euler's magic square

$$L(0, 1, 1, 1, 1, 1, -1, 5) \cdot R(45, -30, -60, 75, 90, 13, -14, -69).$$

Finding a few examples like this required checking thousands of potential integer tuples $(a, b, \dots, h, p, q, r, s, t)$ of length 13. Here we used a combination of backtracking and a greedy algorithm to find tuples $(a, b, \dots, h, p, q, r, s, t)$ of small integers such that the specialized matrix $M \in \mathbb{Q}[u, v, w]^{8\times 8}$ is still proper. For if the integers are large, it is less likely that the system $A = B = 0$ has rational solutions (u, v, w).

3.3. Too strong restrictions. As there are so many more variables than equations, one might consider imposing strong restrictions to make the polynomial system more manageable. For instance, one might pick 3 distinct variables $X, Y, Z \in \{p, q, \ldots, w\}$, and hope to specialize the remaining 13 variables to integers such that $A(X, Y, Z)$ is 0 as a polynomial in X, Y, Z.

For instance, the coefficients in A of w^2 and wp are $8(h-a)(h+a)$ and $16ah$, respectively. So if we want them both to vanish, then $a = h = 0$, in which case M is not proper anymore.

So we cannot have $\{p, w\} \subset \{X, Y, Z\}$. Many other combinations fail for the analogous reason. But some cases require a finer analysis.

3.4. A working restriction. The following compromise proved to be fruitful. We look for conditions on $a, b, \ldots, h \in \mathbb{Z}$ such that A and B both have degree 1 in w. Assume this for a moment. Let x and y be the coefficients of w in A and B, respectively. Then $F = yA - xB$ is a cubic form $F \in \mathbb{Z}[p, q, \ldots, v]$, where we have eliminated w.

Besides having lower degree, the added advantage is that a solution of $F = 0$ with $p, q, \ldots, v \in \mathbb{Q}$ extends to a solution of $A = B = 0$ provided that $x, y \neq 0$.

Fortunately, the condition for A and B to have w-degree ≤ 1 is rather easy and not very restrictive: As noted above, the coefficient of w^2 in A is $8(h-a)(h+a)$. So we need to pick $h = \pm a$. Furthermore, the coefficient of w^2 in B is, up to the factor -2, equal to $b^2+c^2+d^2+e^2+f^2+g^2-3(a^2+h^2) = b^2+c^2+d^2+e^2+f^2+g^2-6a^2$. Thus A and B have w-degree ≤ 1 if and only if $h = \pm a$ and $b^2+c^2+d^2+e^2+f^2+g^2 = 6a^2$.

As remarked previously, if $a = h = 0$, then M is not proper. Thus we assume that

$$h = \pm a \neq 0, \quad b^2+c^2+d^2+e^2+f^2+g^2 = 6a^2. \tag{4}$$

Next let $F = yA - xB$ be the cubic form in $\mathbb{Z}[p, q, \ldots, v]$. The idea is to find a rational specialization of some of the variables $p, q, \ldots, v$ such that F will have degree 1 with respect to one of these variables, because then we get an immediate rational parametrization of $F = 0$ and hence of the solutions of $A = B = 0$.

Note that (4) already implies that F has degree at most 2 in p, because the coefficient of p^3 in F is $32h \cdot (b^2+c^2+d^2+e^2+f^2+g^2-6a^2) = 0$.

Also, the coefficient of p^2 in F has, up to the nonzero factor $-128h^2$, the useful form

$$\begin{aligned}(ag+bh)q + (-af+ch)r + (-ae+dh)s& \\ + (ad+eh)t + (ac+fh)u + (-ab+gh)v&.\end{aligned}$$

As $h = \pm a \neq 0$ and not both b and g vanish, we get $ag + bh \neq 0$ or $-ab + gh \neq 0$. So we can solve for q or v in terms of the remaining variables to find that F has degree at most 1 in p. Usually this degree equals 1, so we can solve for p to make F vanish. Finally, provided that A and B after these substitutions still have degree 1 in w, solving for w finally yields $A = B = 0$, where now the entries of the matrix M are rational functions over $\mathbb{Q}$ in the variables r, s, t, u and q or v.

This is essentially how we get the parametrization in Theorem 1.4, starting with $(a, b, \dots, h) = (2, 1, 1, 4, 2, 1, 1, -2)$.

Remark 3.1. The integer tuples $(a, b, \dots, h)$ satisfying (4) can easily be parametrized. So if we use such a parametrization instead of the specific tuple like $(2, 1, 1, 4, 2, 1, 1, -2)$ which led to Theorem 1.4, we can still carry out the procedure. If we consider matrices over $\mathbb{Q}$ as equivalent if they differ by a nonzero scalar factor, then we obtain an 8-parameter family of proper Euler's magic matrices for which Theorem 1.4 is just a subcase.

4. Improper Euler's $n \times n$ magic matrices for $n \geq 4$. Let σ be an element of the symmetric group on $\{1, \dots, n\}$. We define the matrix $M(\sigma) = (m_{i,j}) \in \mathbb{Z}^{n\times n}$ by

$$m_{i,j} = \begin{cases} 1 & \text{if } j = \sigma(i), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $M(\sigma)$ contains exactly one 1 in each row and column, hence $M(\sigma) \cdot M(\sigma)^t = I_n$.

Theorem 4.1. *Set*

$$\sigma = \begin{cases} (1\ 2\ \dots\ n-1)(n) & \text{if } n \text{ is even,} \\ (1\ 2\ \dots\ k-1)(k)(k+1\ k+2\ \dots n) & \text{if } n = 2k-1 \text{ is odd.} \end{cases}$$

Then $M(\sigma)$ is an Euler's magic matrix.

Proof. We need to verify conditions (1)–(2), and (3). As $M(\sigma) \cdot M(\sigma)^t = I_n$, condition (1) holds with $\gamma = 1$.

First suppose that $n = 2k \geq 4$ is even. Then $m_{i,i} = 0$ for $1 \leq i < n$ and $m_{n,n} = 1$. Furthermore, $m_{i,n+1-i} = 1$ if and only if $i = k$. We see that $M(\sigma)$ fulfills conditions (2) and (3).

Similarly, if $n = 2k-1$ is odd, then $m_{k,k} = 1$ and $m_{i,i} = 0$ if $i \neq k$, and $m_{i,n+1-i} = 1$ if and only if $i = k$. Again, $M(\sigma)$ fulfills conditions (2) and (3). ∎

Of course, these matrices are far from being proper. We have tried to find proper Euler's magic matrices for $n = 5$. The approach was to use the Cayley transform as in the case $n = 3$. This results in two polynomial conditions in 10 variables over $\mathbb{Q}$. Or, after homogenization, we need to solve two polynomials over the integers in 11 unknowns. As all variables appear in a high degree, we basically tried a random search. Surprisingly, this way we obtained quite a few Euler's magic matrices, and some of them came close to properness. In fact, the following are Euler's magic matrices, where the squares of its entries give 24 distinct elements. In each case we highlight the pair of entries which violates properness:

$$\begin{pmatrix} -106 & -32 & -8 & -75 & -50 \\ -4 & -38 & -120 & 58 & -35 \\ 24 & \mathbf{20} & -73 & -88 & 80 \\ 61 & 66 & -16 & -46 & -100 \\ 70 & -115 & \mathbf{20} & -40 & -18 \end{pmatrix},$$

$$\begin{pmatrix} 4 & 3 & 40 & -94 & -142 \\ -29 & -128 & -90 & 44 & -58 \\ 154 & 28 & -35 & 56 & -42 \\ 74 & -\mathbf{82} & -10 & -114 & 73 \\ -24 & \mathbf{82} & -140 & -61 & 2 \end{pmatrix},$$

$$\begin{pmatrix} -204 & -38 & 10 & -11 & -312 \\ 54 & -262 & -260 & 36 & -13 \\ -84 & \mathbf{102} & -165 & -306 & 48 \\ 291 & \mathbf{102} & -40 & -56 & -202 \\ 66 & -223 & 210 & -206 & -2 \end{pmatrix},$$

$$\begin{pmatrix} 29 & -218 & -370 & -\mathbf{188} & 180 \\ 88 & -384 & 22 & 158 & -269 \\ -160 & 58 & -40 & -333 & -334 \\ 210 & -139 & 304 & -286 & 124 \\ 418 & \mathbf{188} & -147 & -4 & -146 \end{pmatrix},$$

$$\begin{pmatrix} -\mathbf{392} & -336 & -21 & 282 & 210 \\ 177 & -384 & -24 & -\mathbf{392} & 240 \\ 408 & -186 & -246 & 357 & -40 \\ -48 & -309 & 176 & -42 & -510 \\ -192 & 14 & -546 & -168 & -165 \end{pmatrix}.$$

A verification of these examples is again provided in [9].

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