A note on primitive groups containing a 3–cycle

Theorem 1. *Let Ω be a (possibly infinite) set and $G \leq \mathrm{Sym}(\Omega)$ be a group acting primitively on Ω. If G contains a 3–cycle, then G contains all 3–cycles of $\mathrm{Sym}(\Omega)$.*

This theorem is well known for finite Ω, implying that in this case $G = \mathrm{Alt}(\Omega)$ or $G = \mathrm{Sym}(\Omega)$. Unfortunately, the usual proofs obtain this result as a consequence of more difficult theorems by Jordan. Here we give a simple proof, which works without any change in the infinite case as well.

Lemma 2. *Let $\sigma = (a\,b\,c)$ and $\tau = (d\,e\,f)$ be 3–cycles in $\mathrm{Sym}(\Omega)$. Set $\Delta = \{a, b, c, d, e, f\}$ and consider $H = <\sigma, \tau>$ as a subgroup of $\mathrm{Sym}(\Delta)$. If $|\Delta| \leq 5$, then H contains all 3–cycles from $\mathrm{Sym}(\Delta)$.*

Proof. As $\{a, b, c\}$ and $\{d, e, f\}$ are not disjoint, H acts transitively on Δ, hence $|\Delta|$ divides $|H|$. Furthermore, 3 divides $|H|$.

If $|\Delta| = 3$, then the assertion is clear. Next assume $|\Delta| = 4$. Then 12 divides $|H|$. On the other hand, $H \leq \mathrm{Alt}(\Delta) \cong \mathrm{Alt}_4$, so $H = \mathrm{Alt}(\Delta)$.

Finally assume $|\Delta| = 5$, so 15 divides $|H|$. We may assume that $c = d$, so a, b, c, e, f are pairwise distinct. We compute

$$\sigma \cdot \sigma^{\tau} = (a\,b\,c) \cdot (a\,b\,c)^{(c\,e\,f)} = (a\,b\,c) \cdot (a\,b\,e) = (a\,e)(b\,c),$$

so $|H|$ is even and therefore 30 divides $|H|$. So H has index at most 2 in $\mathrm{Alt}(\Delta)$, and is normal in this group. So all elements of order 3 from $\mathrm{Alt}(\Delta)$ are contained in H. $\square$

Lemma 3. *We define a relation $\sim$ on Ω as follows: $a \sim a$ for all a; and if $a \neq b$, then $a \sim b$ if and only if there is c distinct from a and b with $(a\,b\,c) \in G$. Then $\sim$ is a G–invariant equivalence relation.*

Proof. The only non–trivial property to verify is the transitivity of this relation. For this assume that a, b, c are pairwise distinct, and $a \sim b$ and $b \sim c$. So there are d and e with $\sigma = (a\,b\,d)$ and $\tau = (b\,c\,e)$ in G. By the previous lemma, $<\sigma, \tau>$ contains $(a\,b\,c)$, hence $a \sim c$. $\square$

We now prove the theorem. As G is primitive on Ω, and the equivalence classes of a G–invariant equivalence relation $\sim$ is a block system, we obtain that $\sim$ is one of the two trivial relations on Ω. However, as G contains a 3–cycle, there are distinct a, b with $a \sim b$. So $a \sim b$ for all a and b.

Let a, b, and c be pairwise distinct. We therefore obtain $d \in \Omega \setminus \{a, b\}$ and $e \in \Omega \setminus \{b, c\}$ such that the 3–cycles $\sigma = (a\,b\,d)$ and $\tau = (b\,c\,e)$ are contained in G. Again by Lemma 2, we get $(a\,b\,c) \in G$.

E-mail: `peter.mueller@mathematik.uni-wuerzburg.de`