A REMARK ON THE SUM OF SQUARES LAW

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The main result of the paper [1]"The Sum of Squares Law" by Julio Kovacs, Fang Fang, Garrett Sadler, and Klee Irwin is based on the following theorem, applied to a proper orthogonal projection π and the trivial projection $\pi = I_n$, respectively. However, the proof given there seems to use the wrong assertion that an irreducible real representation remains irreducible over the complex numbers. Furthermore, it has the superfluous restriction to proper symmetries.

Thus we give a different version and proof of this key assertion.

Theorem. *Let $G \leq O_n(\mathbb{R})$ be a finite, irreducible group of orthogonal matrices in their natural action on $\mathbb{R}^n$, and let $\pi \in \mathbb{R}^{n\times n}$ be a matrix such that $\boldsymbol{v} \mapsto \pi\boldsymbol{v}$ is an orthogonal projection to an m-dimensional subspace. Then*

$$\sum_{g\in G} \|\pi g\boldsymbol{v}\|^2 = \frac{m}{n}|G|\|\boldsymbol{v}\|^2$$

for all $\boldsymbol{v} \in \mathbb{R}^n$.

Proof. Denoting $\langle\cdot,\cdot\rangle$ the standard scalar product of $\mathbb{R}^n$, we get

$$\sum_{g\in G} \|\pi g\boldsymbol{v}\|^2 = \sum_{g\in G} \langle \pi g\boldsymbol{v}, \pi g\boldsymbol{v}\rangle = \sum_{g\in G} (\pi g\boldsymbol{v})^t(\pi g\boldsymbol{v}) = \boldsymbol{v}^t(\sum_{g\in G} g^t\pi^t\pi g)\boldsymbol{v}.$$

Set $\phi = \sum_{g\in G} g^t\pi^t\pi g$. Note that $\phi^t = \phi$. Furthermore, $h^t\phi h = \phi$ for all $h \in G$, since if g runs through G, then so does gh. From $h^t = h^{-1}$ we see that ϕ commutes with all $h \in G$.

The eigenvalues of the symmetric matrix ϕ are real. Let λ be an eigenvalue. Then $\phi - \lambda I_n$ is singular. On the other hand, $\phi - \lambda I_n$ commutes with G, so by Schur's Lemma, $\phi - \lambda I_n$ is either 0 or invertible. Thus $\phi = \lambda I_n$.

Taking the trace in

$$\lambda I_n = \sum_{g\in G} g^t\pi^t\pi g = \sum_{g\in G} g^{-1}\pi^t\pi g$$

shows $n\lambda = |G| \operatorname{trace}(\pi^t\pi)$.

Note that π is orthogonally conjugate to a diagonal matrix with m entries 1 on the diagonal, and the remaining entries 0. Thus $\operatorname{trace}(\pi^t\pi) = m$, so $n\lambda = m|G|$, and the claim follows. □

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[1]https://arxiv.org/abs/1210.1446