COMBINING A HEDGEHOG OVER $\mathbb{Q}_2$

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Theorem. *There is a* 3×3 *matrix with entries in* $\mathbb{Z}_2[X,Y,Z]$ *and first row* $[X\ Y\ Z]$ *such that its determinant is* 1 *modulo* $X^2+Y^2+Z^2-1$.

Proof. Let ν be the 2-adic valuation of $\mathbb{Q}_2$. The two roots of T^2-T+2 have values 0 and 1, respectively. Let ω be the root with $\nu(\omega)=0$.

Let M be the matrix

$$\begin{bmatrix} X & Y & Z \\ -Y+Z+1 & (1-\omega)X+Y+(1+\omega)Z+\omega & -X-Y+2Z+(1-\omega) \\ \omega Y+(2-\omega)Z & (1-2\omega)X+(1+\omega)Y+3Z+1 & -2X+(2-\omega)Z-\omega \end{bmatrix}.$$

A lengthy but straightforward calculation (but see the remark below) shows that

$$\det M-(2-\omega)=-(X+\omega Y+Z+(2-\omega))(X^2+Y^2+Z^2-1).$$

Clearly the matrix M has entries in $\mathbb{Z}_2[X,Y,Z]$. Note that $2-\omega$ is a unit in $\mathbb{Z}_2$ as $\nu(2)=1\neq 0=\nu(\omega)$. Thus, dividing the third row of M by $2-\omega$, the entries stay in $\mathbb{Z}_2[X,Y,Z]$ and the resulting matrix has determinant 1 modulo $X^2+Y^2+Z^2-1$. □

Remark. If one does not want to do the calculation by hand, the following SageMath code verifies this example too. From most pdf viewers (like evince) one can copy the following code and paste it in a SageMath console or the online calculator at `https://sagecell.sagemath.org/`.

```
t = polygen(QQ)
K.<w> = NumberField(t^2-t+2)
R.<x, y, z> = K[]
M = matrix(3, 3, [x, y, z, -y + z + 1,
                  (-w + 1)*x + y + (w + 1)*z + w,
                  -x - y + 2*z + (-w + 1),
                  w*y + (-w + 2)*z,
                  (-2*w + 1)*x + (w + 1)*y + 3*z + 1,
                  -2*x + (-w + 2)*z + (-w)])
print(M.det() - (2-w) ==
      - (x + w*y + z + (w - 2)) * (x^2 + y^2 + z^2 - 1))
```

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