On products of transpositions

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We give another proof of the following well-known result:

Theorem. *Let $\sigma_1, \dots, \sigma_r$ be transpositions on the set Ω. If $\sigma_1\sigma_2\cdots\sigma_r = 1$, then r is even.*

Proof. We let the symmetric group $\mathrm{Sym}(\Omega)$ act from the right on Ω. Consider a counter example with r minimal, and pick $\omega \in \Omega$ such that ω is moved by σ_1. Replacing two consecutive transpositions α, β in the product by the transpositions β, α^β preserves r and, as $\alpha\beta = \beta\alpha^\beta$, the product 1 property. After a finite number of such replacements we may and do assume that there is $k \geq 1$ such that each of $\sigma_1, \sigma_2, \dots, \sigma_k$ moves ω, while σ_i fixes ω for $i > k$. Hence

$$\omega^{\sigma_{k+1}\sigma_{k+2}\cdots\sigma_r} = \omega = \omega^{\sigma_1\sigma_2\cdots\sigma_r},$$

and therefore

$$\omega = \omega^{\sigma_1\sigma_2\cdots\sigma_k}.$$

As $\omega \neq \omega^{\sigma_1}$, there must be an element σ_i with $2 \leq i \leq k$ which moves ω^{σ_1}. As it moves ω too, we get $\sigma_1 = \sigma_i$. Use the replacements from above to bring σ_i into the second position. So σ_1 and σ_i cancel out, contrary to the minimal choice of r. □

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