A NOTE ABOUT INCIDENCE MATRICES OF PROJECTIVE PLANES

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A finite group G of order $n^2 + n + 1$ is a Singer group of a finite projective plane if and only if G contains a difference set D of size $n + 1$. If we consider the regular permutation action of G, then the latter is equivalent to $\{gh^{-1} \mid g, h \in D, g \neq h\} \cup \{e\}$ being a sharply transitive set of permutations.

The following theorem shows that an arbitrary finite projective plane is in some sense a generalization of the Singer planes.

Theorem. *There is a projective plane of order n if and only if the symmetric group S_{n^2+n+1} contains a subset D of size $n+1$ such that $\{gh^{-1} \mid g, h \in D, g \neq h\} \cup \{e\}$ is a sharply transitive set of permutations.*

Proof. Let A be the incidence matrix of a projective plane of order n, and I and J be the identity and the all-1-matrix of size $n^2 + n + 1$, respectively.

An easy application of Hall's marriage theorem [1] shows that A is a sum of $n + 1$ permutation matrices. Let D be the corresponding permutations in S_{n^2+n+1}. Then $AA^t = nI + J$ together with $P^t = P^{-1}$ for permutation matrices P translates to the claim. □

Remark. The theorem has a distant similarity with Witt's theorem that there is a projective plane of order n if and only if S_n contains a sharply 2-transitive set of permutations.

Nevertheless, none of these theorem seems to be usable to reprove the Bruck-Ryser non-existence theorem or subcases thereof.

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[1]As in the standard proof of Birkhoff's theorem that every doubly stochastic matrix is a convex combination of permutation matrices.