Theorem. $\varphi(n) > n^{2/3}$ *for all* $n > 64$.

Proof. Let $n = \prod_{i=1}^{m-1} {p_i}^{e_i}$ with $m-1$ distinct prime numbers $p_1 < p_2 < \cdots < p_{m-1}$ and $e_i \geq 1$. Clearly $p_i \geq i+1$, hence

$$\frac{\varphi(n)}{n} = \prod_{i=1}^{m-1}(1-\frac{1}{p_i}) \geq \prod_{i=1}^{m-1}(1-\frac{1}{i+1}) = \prod_{i=1}^{m-1}\frac{i}{i+1} = \frac{1}{m}.$$

Suppose that $\varphi(n) \leq n^{2/3}$. Then $n \leq m^3$. On the other hand

$$n \geq \prod_{i=1}^{m-1} p_i \geq \prod_{i=1}^{m-1}(i+1) = m!,$$

hence

$$m! \leq n \leq m^3.$$

By induction, we see that $m! \leq m^3$ implies $m \leq 5$. If $m \leq 4$, then $n \leq 64$, and were done. Thus assume $m = 5$, so $n \leq 5^3 = 125$. On the other hand, n has $4 = m-1$ distinct prime divisors, hence $n \geq 2 \cdot 3 \cdot 5 \cdot 7 = 210$, a contradiction. □

Remark. (a) Actually $\varphi(n) > n^{2/3}$ for all n except $2, 3, 4, 6, 10, 12, 18, 24, 30, 42$.

(b) The given method shows that for each $0 < \alpha < 1$ there is a bound K_α such that $\varphi(n) > n^\alpha$ for all $n \geq K_\alpha$. For instance, if $n > (2k)^k$ for $k \in \mathbb{N}$, then $\varphi(n) > n^{1-\frac{1}{k}}$. Following the argument from above, this reduces to showing that $m! > m^k$ for $m > 2k$. But that can be seen as follows: For $1 \leq i \leq m$ we have $i(m+1-i) \geq m$. Multiplying these m inequalities gives $m!^2 \geq m^m > m^{2k}$ as required.