COMMON EIGENVECTOR OF COMMUTING PAIR OF ENDOMORPHISMS

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Theorem. *Let V be a finite dimensional vector space over a field K, such that every endomorphism of V has an eigenvalue in K. Then any two commuting endomorphisms of V have a common eigenvector in V.*

Proof. We prove the claim by induction on $\dim V$. There is nothing to do if $\dim V = 1$.

Now suppose that $\dim V > 1$, and let $a, b \in \operatorname{End} V$ be two commuting endomorphisms. Let λ be an eigenvalue of a, and U be the kernel of $a - \lambda\mathrm{id}$. Since $b(U) \subseteq U$, we are done if b has an eigenvector in U. Suppose that this is not the case.

Let W be the image of $a - \lambda\mathrm{id}$, and U' be a complement of W in V. As $\dim U = \dim U'$, there is an automorphism c of V such that $c(U) = U'$. Note that $c \circ b \circ c^{-1} \in \operatorname{End} U'$. Let $d \in \operatorname{End} W$ be arbitrary. We claim that d has an eigenvector in W. To see this, we define $d' \in \operatorname{End} V = \operatorname{End}(U' \oplus W)$ by

$$d'(u' + w) := c(b(c^{-1}(u'))) + d(w).$$

As b has no eigenvector in U, neither has d' in U'. On the other hand, d' has an eigenvector in $V = U' \oplus W$ by the assumption, so d has an eigenvector in d.

From $a(W) \subseteq W$ and $b(W) \subseteq W$ and $\dim W < \dim V$ and the induction hypothesis we obtain a common eigenvector of a and b in W. □

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