

ALMOST SURJECTIVE POLYNOMIAL MAPS

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In 2009 Philipp Lampe asked one of the most popular questions on MathOverflow [3]:

Can a non-surjective polynomial map from an infinite field to itself miss only finitely many points?

Despite considerable efforts, this question remained open for 16 years. For many fields K the answer is easily seen to be negative. Of course, this is the case if K is algebraically closed. It is also the case if K is the field of real numbers, because if the polynomial misses a value a , it also omits a neighborhood of a . For the same reason, the answer is negative for p -adic fields.

More generally, Michiel Kusters proved that the answer is negative if K is a perfect large¹ field, see [2].

All these fields are in a sense close to being algebraically closed.

At the other extreme, the answer is also negative for number fields or more generally for Hilbertian fields by an easy application of Hilbert's Irreducibility Theorem.

However, in [1, Theorem D], Will Johnson, Chieu-Minh Tran, Erik Walsberg, and Jinhe Ye proved that the answer is positive for certain imperfect PAC fields² K . Their proof, which uses model theory, is quite involved. In [4], one author of [1] describes a more direct construction, using infinite Galois theory and Sylow theory of profinite groups.

In this note we give an even shorter and completely elementary construction.

Theorem. *For each odd prime p there exists a field K of characteristic p and $t \in K$ such that $f(K) = K \setminus \{0\}$ for $f(X) = (X^2 - t)(X^p - t)$.*

Proof. Let t be a transcendental over $\mathbb{F}_p$ and S be a separable closure of $\mathbb{F}_p(t)$. By Zorn's Lemma, there exists a field K with $\mathbb{F}_p(t) \subseteq K \subseteq S$ which is maximal with respect to $\sqrt{t} \notin K$. Thus every proper finite separable extension L of K contains $\sqrt{t}$ and therefore $[L : K]$ is even.

¹A field K is called large (also called ample) if every irreducible K -curve with a K -rational smooth point has infinitely many K -rational points.

²A field K is called a PAC field if every absolutely irreducible plane algebraic K -curve has infinitely many K -rational points.

Set $f(X) = (X^2 - t)(X^p - t)$. Pick $a \in K$. The derivative of $f(X)$ is $f'(X) = 2X(X^p - t)$. Thus a non-constant common factor of $f(X) - a$ and its derivative exists only for $a = 0$ and for $a = t^2$.

Suppose that $a \neq 0$ and $a \neq t^2$, so $f(X) - a$ is separable. Then each irreducible factor of $f(X) - a$ is either linear or has even degree. But $f(X) - a$ has odd degree $p + 2$, so $f(X) - a$ has a root in K and therefore $a \in f(K)$.

As to the case $a = t^2$, note that $t^2 = f(0) \in f(K)$.

Furthermore, $X^p - t$ has no root in K (not even in S), and neither has $X^2 - t$ by construction of K . So $0 \notin f(K)$. $\square$

Remark. A similar construction yields an example for $p = 2$. Let S be a separable closure of $\mathbb{F}_4(t)$, and $\mathbb{F}_4(t) \subseteq K \subseteq S$ with K maximal subject to $\theta \notin K$, for a fixed $\theta \in S$ with $\theta^3 = t$. Then arguments analogous to those above show $f(K) = K \setminus \{0\}$ for $f(X) = (X^2 - t)(X^3 - t)$.

REFERENCES

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